Some notes on the Riemann integral

This section provides an alternate approach to the Riemann integral. The goal is to show that monotone functions are Riemann integrable. The definition of upper and lower sums can be found in the note at the bottom of page 299 of Stewart. Observe, however, that this only works for continuous functions since for more general functions the infimum and supremum might not be obtained on the intervals $[x_i, x_{i+1}]$.

Definition 1. *A partition P of the interval (a,b) is a finite subset of (a,b) containing both a and b. We can order the set P in increasing order as $P = \{p_0, p_1, p_2, \ldots, p_n\}$. In other words:*

- $p_0 = a$
- $p_n = b$
- $p_j < p_{j+1}$ *if* $0 \leq j < n$.

The infimum of a set can be defined analogously to the supremum, but the following also works.

Definition 2. *If $A \subseteq \mathbb{R}$ then t is the infimum of A if $-t$ is the supremum of $-A = \{-a \mid a \in A\}$.*

Definition 3. *If f is a function from the interval (a,b) to the interval (c,d) and $P = \{p_0, p_1, p_2, \ldots, p_n\}$ is a partition of (a,b) and $1 \leq j \leq n$ define:*

- *m_j to be the infimum of $\{f(x) \mid p_{j-1} \leq x \leq p_j\}$*
- *M_j to be the supremum of $\{f(x) \mid p_{j-1} \leq x \leq p_j\}$.*

If we restrict to continuous functions then we could use the text's definition:

- *m_j to be the minimum value of $\{f(x) \mid p_{j-1} \leq x \leq p_j\}$*
- *M_j to be the maximum value of $\{f(x) \mid p_{j-1} \leq x \leq p_j\}$.*

The m_j and M_j depend on f and P of course, but it will usually not cause problems to supress this in the notation. However, if it is necessary to indicate the dependence on P the notation $m_j(P)$ and $M_j(P)$ will be used.

Definition 4. *If f is a function from the interval (a,b) to the interval (c,d) and $P = \{p_0, p_1, p_2, \ldots, p_n\}$ is a partition of (a,b) define*

- $U(f,P) = \sum_{j=1}^{n} M_j(p_j - p_{j-1})$
- $L(f,P) = \sum_{j=1}^{n} m_j(p_j - p_{j-1})$.

Note that if f is a positive function then $U(f,P)$ is the sum of the areas of rectangles whose union contains the region bounded by the graph of f, while $L(f,P)$ is the sum of the areas of rectangles whose union is contained in the region bounded by the graph of f.

Definition 5. *If f is a function from the interval (a,b) to the interval (c,d) define*

- *$L(f)$ to be the supremum of the set of all real numbers $L(f,P)$ where P is a partition of (a,b)*
- *$U(f)$ to be the infimum of the set of all real numbers $U(f,P)$ where P is a partition of (a,b).*

Definition 6. *If f is a function from the interval (a,b) to the interval (c,d) and $U(f) = L(f)$ then f will be said to be Riemann integrable and the common value $U(f) = L(f)$ is defined to be $\int_a^b f$.*

Lemma 1. *If f is a function from the interval (a,b) to the interval (c,d) and P is a partition of (a,b) then $U(f,P) \geq L(f,P)$.*

Proof. It suffices to show $U(f,P) - L(f,P) \geq 0$. From Definition 4 we know that $U(f,P) = \sum_{j=1}^{n} M_j(p_j - p_{j-1})$ and $L(f,P) = \sum_{j=1}^{n} m_j(p_j - p_{j-1})$ so

$$U(f,P) - L(f,P) = \sum_{j=1}^{n} M_j(p_j - p_{j-1}) - \sum_{j=1}^{n} m_j(p_j - p_{j-1}) = \sum_{j=1}^{n} (M_j - m_j)(p_j - p_{j-1})$$

Since $p_j > p_{j-1}$ for each j it follows that $(p_j - p_{j-1}) \geq 0$ for each j. Hence it suffices to show that $M_j \geq m_j$. From Definition 3 we know that m_j to be the infimum of $\{f(x) \mid p_{j-1} \leq x \leq p_j\}$ and M_j is the supremum of the same set. Hence $M_j \geq m_j$. □

Lemma 2. *If f is a function from the interval (a,b) to the interval (c,d) and P and Q are partitions of (a,b) and $P \subseteq Q$ then*

- $U(f,Q) \leq U(f,P)$
- $L(f,Q) \geq L(f,P)$.

Proof. It will only be shown that $U(f,Q) \leq U(f,P)$, the other assertion being similar and left as an exercise. Consider first the case that Q has precisely one more point than P. Suppose that $P = \{p_0, p_1, \ldots p_n\}$ and $Q = P \cup \{p^*\}$. There is then some j such that $p_{j-1} < p^* < p_j$. Then

$$U(f,P) = M_1(p_1 - p_0) + M_2(p_2 - p_1) + \ldots + M_j(p_j - p_{j-1}) + \ldots + M_n(p_n - p_{n-1})$$

and

$$U(f,Q) = M_1(p_1 - p_0) + M_2(p_2 - p_1) + \ldots + M_j(Q)(p^* - p_{j-1}) + M_{j+1}(Q)(p_j - p^*) + \ldots + M_n(p_n - p_{n-1})$$

where the explicit dependence of $M_j(Q)$ and $M_{j+1}(Q)$ is indicated by the notation. Note that $M_n(P) = M_n(Q)$ if $n < j-1$ and $M_n(P) = M_{n+1}(Q)$ if $n > j$.

Hence it suffices to show that

$$M_j(p_j - p_{j-1}) \geq M_j(Q)(p^* - p_{j-1}) + M_{j+1}(Q)(p_j - p^*).$$

But

$$M_j(p_j - p_{j-1}) = M_j(p^* - p_{j-1}) + M_j(p_j - p^*)$$

and so it suffices to show that $M_j \leq M_j(Q)$ and $M_j \leq M_{j+1}(Q)$. But according to Definition 3,

$$M_j \text{ is the supremum of } \{f(x) \mid p_{j-1} \leq x \leq p_j\}$$

whereas

$$M_j(Q) \text{ to be the supremum of } \{f(x) \mid p_{j-1} \leq x \leq p^*\}$$

and so $M_j(Q) \leq M_j$ since it is the supremum taken over a smaller set. Similary,

$$M_{j+1}(Q) \text{ to be the supremum of } \{f(x) \mid p^* \leq x \leq p_j\}$$

and so $M_{j+1}(Q) \leq M_j$ since it is also the supremum taken over a smaller set.

The general result then follows by induction on the number of elements in Q that are not in P. This is left as an exercise. □

Corollary 1. *If f is a function from the interval (a,b) to the interval (c,d) and P and Q are partitions of (a,b) then $L(f,P) \leq U(f,Q)$.*

Proof. Using Lemma 1 and Lemma 2

$$L(f,P) \leq L(f,P \cup Q) \leq U(f,P \cup Q) \leq U(f,Q).$$

□

Corollary 2. *If f is a function from the interval (a,b) to the interval (c,d) then $L(f) \leq U(f)$.*

Proof. $L(f)$ is the supremum of the set of all $L(f,P)$ while $U(f)$ is the infimum of the set of all $U(f,P)$. Since each $L(f,P)$ is less than or equal to any $U(f,Q)$ the result follows. □

Corollary 3. *If f is a function from the interval (a,b) to the interval (c,d) then in order to show that f is Riemann integrable it suffices to show that for any $\epsilon > 0$ there is some partition P such that $U(f,P) - L(f,P) < \epsilon$.*

Proof. Proceed by contradiction. Assume that $U(f) \neq L(f)$. By Corollary 2 it follows that

$$\epsilon = U(f) - L(f) > 0. \tag{1}$$

Then use the hypothesis to find a partition P such that $U(f,P) - L(f,P) < \epsilon$. Then $U(f,P) \geq U(f)$ and $L(f,P) \leq L(f)$. Hence

$$U(f) - L(f) \leq U(f,P) - L(f,P) < \epsilon = U(f) - L(f)$$

using (1) and this is a contradiction. □

Theorem 1. *If f is a monotone function from (a,b) to (c,d) then f is Riemann integrable.*

Proof. Assume f is non-decreasing as a similar proof works for the case that f is non-increasing. Using Corollary 3 let $\epsilon > 0$. Let n be an integer so large that

$$\frac{(d-c)(b-a)}{n} < \epsilon \tag{2}$$

or, in other words, choose n to be an integer larger than $(d-c)(b-a)/\epsilon$.

Now let $s = (b-a)/n$ and let P be the partition defined by

$$P = \{a, a+s, a+2s, a+3s, \ldots, a+(n-1)s, a+ns\}$$

and note that $a + ns = a + (b-a) = b$. Moreover, the points of P are equally spaced with consecutive points a distance s apart. If we write $P = \{p_0, p_1, p_2, \ldots, p_n\}$ then $p_j = a + js$ for each j between 0 and n. In other words,

$$p_0 = a, p_1 = a+s, p_2 = a+2s, \ldots p_j = a+js, \ldots p_n = a+ns = b.$$

Notice that since f is non-decreasing, it follows that if $p_{j-1} \leq x \leq p_j$ then $f(p_{j-1}) \leq f(x) \leq f(p_j)$ for each j between 1 and n. Hence $m_j = f(p_{j-1})$ and $M_j = f(p_j)$ for each such j. This implies that

$$\text{if } j > 1 \text{ then } m_j = M_{j-1}. \tag{3}$$

Notice also that since the p_j are equally spaced, it follows that $p_j - p_{j-1} = (a+js) - (a+(j-1)s) = s$ for each j. It follows that

$$U(f,P) = \sum_{j=1}^{n} M_j(p_j - p_{j-1}) = \sum_{j=1}^{n} M_j s = \sum_{j=1}^{n-1} M_j s + M_n s$$

and

$$L(f,P) = \sum_{j=1}^{n} m_j(p_j - p_{j-1}) = \sum_{j=1}^{n} m_j s = m_1 s + \sum_{j=2}^{n} m_j s = m_1 s + \sum_{j=2}^{n} M_{j-1} s = m_1 s + \sum_{j=1}^{n-1} M_j s$$

where (3) was used for the second to last equality. It follows that

$$U(f,P) - L(f,P) = M_n s - m_1 s = (M_n - m_1)s$$

and, noting that $M_n \leq d$ and $m_1 \geq c$ it follows that

$$U(f,P) - L(f,P) \leq (d-c)s = (d-c)(b-a)/n < \epsilon$$

as required. □